Barrier Options

Arfima Financial Solutions

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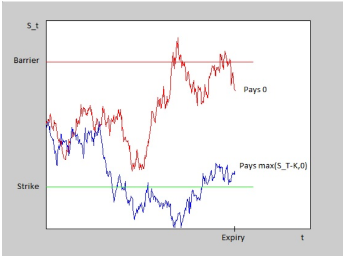
Definition

- **Definition:** A barrier option is an option on the underlying asset that is activated or extinguished in the underlying reaches a predetermined level (the barrier).
- **Types:**
  - Option type: call or put.
  - Barrier and underlying level: down (the initial level of the underlying is above the barrier) or up (the initial level of the underlying is below the barrier).
  - Activated or extinguished: knock-in (the option is activated if the underlying reaches the barrier) or knock-out (the option is extinguished if the underlying reaches the barrier).
  - Exercise: European, Bermudan, American.

Example

- **European up & out call:** is an European call that is extinguished if the underlying crosses the barrier before expiration. The initial value of the underlying is below the barrier.
- **Payoff:**

$$X_T^{\mathrm{U\&O}} = \begin{cases} \max(S_T - K, 0) & \text{if } S_t < B \text{ for all } t \le T \\ 0 & \text{otherwise} \end{cases}$$



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Pricing under Black-Scholes framework. Up & out case.

- Under the risk-neutral measure the underlying is a geometric Brownian motion
$$dS_t = rS_t dt + \sigma S_t dW_t.$$

- **Key result [1]:** The joint density of a Brownian motion W_T and its maximum $M_T = \max_{0 \le t \le T} W_t$ is

$$f_{M_T,W_T}(m,w) = \frac{2(2m-w)}{T\sqrt{2\pi T}} \exp\left(-\frac{1}{2T}(2m-w)^2\right), \qquad w \le m, \quad m \ge 0,$$

and zero for other values of m and w.

Pricing under Black-Scholes framework. Up & out case.

- **Partial differential equation approach:** Assuming $B > K$, the value $V(t,x)$ at time t of an up & out European call with expiry T subject to $S(t) = x$ satisfies the PDE

$$\frac{\partial V}{\partial t} + rx\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} = rV,$$

with the boundary conditions

$$\begin{aligned} V(t,0) &= 0, && 0 \le t \le T, \\ V(t,B) &= 0, && 0 \le t < T, \\ V(T,x) &= \max(x-K,0), && 0 \le x \le B, \end{aligned}$$

where K is the strike and B is the barrier.

Pricing under Black-Scholes framework. Up & out case.

- **Analytic formula:**

$$V(t,x) = xI_1 - KI_2 - xI_3 + KI_4, \tag{1}$$

where

$$I_1 = \Phi\left(d_+\left(T-t, \frac{x}{K}\right)\right) - \Phi\left(d_+\left(T-t, \frac{x}{B}\right)\right),$$

$$I_2 = e^{-r(T-t)}\left[\Phi\left(d_-\left(T-t, \frac{x}{K}\right)\right) - \Phi\left(d_-\left(T-t, \frac{x}{B}\right)\right)\right],$$

$$I_3 = \left(\frac{x}{B}\right)^{-\frac{2r+\sigma^2}{\sigma^2}}\left[\Phi\left(d_+\left(T-t, \frac{B^2}{xK}\right)\right) - \Phi\left(d_+\left(T-t, \frac{B}{x}\right)\right)\right],$$

$$I_4 = e^{-r(T-t)}\left(\frac{x}{B}\right)^{-\frac{2r-\sigma^2}{\sigma^2}}\left[\Phi\left(d_-\left(T-t, \frac{B^2}{xK}\right)\right) - \Phi\left(d_-\left(T-t, \frac{B}{x}\right)\right)\right],$$

with

$$d_\pm(\tau, s) = \frac{1}{\sigma\sqrt{\tau}}\left(\log s + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right),$$

and Φ the standard normal probability function.

Pricing under Black-Scholes framework. Up & out case.

- The other cases (down & out, up & in and up & out) are analogous.
- **In-out parity:** Holding a knock-in and a knock-out barrier option is equivalent to holding the corresponding vanilla option.
- Other models without analytic formula:
  - Constant elasticity of variance: $\sigma(t, S_t) = \sigma S_t^{\gamma}$
  - Stochastic volatility: Heston model.
  - Local volatility: Dupire's equation.

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Risk management

- The risk management of this kind of options is carried out computing the sensitivity of the value of the option to changes in the parameters, that is, computing the Greeks.
- Greeks: Delta (sensitivity to underlying value), Gamma (speed of changing in the value of the option with respect to changes in the value of the underlying), Vega (sensitivity to volatility), Rho (sensitivity to interest rates).
- Computation of the Greeks by Monte Carlo schemes is numerically unstable.
- Under Black-Scholes framework Greeks can be computed directly deriving formula (1).
- Remark: Infinite Gamma at the barrier $\Rightarrow$ impossible to delta-hedge.

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Second generation barrier options

- **Digital Barrier:** Barrier options with rebate.
- **Double Barrier:** The option is activated or extinguished if the underlying reaches an upper or a lower barrier.
- **External Barrier:** The payoff and the barrier are defined on different underlyings.
- **Parisian:** The option is activated or extinguished if the underlying stays a certain amount of time beyond the barrier.

S.E. Shreve, *Stochastic Calculus for Finance II: Continuous time models*, Springer, 2004.